Integrals of $x^{(x^n)}$ and $x^{-(x^n)}$ from 0 to 1

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The integral of the function x^x from 0 to 1 does not have a closed form. Likewise, the integral of the function x^{-x} from 0 to 1 does not have a closed form. They can be expressed as convergent series.

The function x^x

Observe that x^x is not defined at $x = 0$. For $x > 0$, we define x^x via the exponential function by $x^x = \exp(x\ln(x)) = e^{x\ln(x)}$. Now, using L' Hôpital's Rule, $\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}(-x) = 0$. Define $f(x) = \begin{cases} x\ln(x), & x > 0, \\ 0, & x = 0 \end{cases}$ on the closed interval [0, 1]. Then f is continuous on [0, 1].

Note that $f(x) \le 0$ for x in [0, 1]. Moreover, $f'(x) = \ln(x) + 1$ for $x > 0$. Hence, $f'(x) = 0$ in (0, 1] if, and only if, $x = e^{-1}$. Therefore, $f'(x) < 0$ for $x \in (0, e^{-1})$ and $f'(x) > 0$ for $x \in (e^{-1}, 1]$. Consequently, $f(x)$ is decreasing on $[0, e^{-1}]$ and increasing on $[e^{-1}, 1]$. Hence, the function f has an absolute minimum value of $-e^{-1}$ on [0, 1] and a maximum value of 0 at 0 and 1. Let $g(x) = e^{f(x)}$ for $x \in [0,1]$. Then $g(x)$ is a continuous function on [0, 1], since it is a composite of two continuous functions. Thus, by x^x for $x \in [0,1]$, we mean $g(x)$. Hence, x^x is Riemann integrable on [0, 1] and $\int_0^1 x^x dx = \int_0^1 g(x)dx = \int_0^1 e^{f(x)}dx$. Note that $e^{(-e^{-1})} \le g(x) \le 1$ for $x \in [0,1]$.

We can use the power series expansion of the exponential function to express $g(x)$ too.

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(f(x))^n}{n!} .$$

Let $g_n(x) = 1 + \sum_{k=1}^{n} \frac{(f(x))^k}{k!}$ for integer $n \ge 1$. Then $g_n(x)$ converges pointwise to $g(x)$ on [0, 1].

Moreover, $|g_n(x)| \le 1 + \sum_{k=1}^{n} \frac{(-f(x))^k}{k!} \le 1 + \sum_{k=1}^{n} \frac{(e^{-1})^k}{k!} \le 1 + \sum_{k=1}^{\infty} \frac{1}{k!} = e$. This means that the sequence of functions $(g_n(x))$ is uniformly bounded on [0, 1]. Therefore, by the Arzelà's Dominated Convergence Theorem,

$$\int_0^1 g_n(x)dx = 1+\sum_{k=1}^{n}\int_0^1 \frac{(f(x))^k}{k!}dx \to \int_0^1 g(x)dx .$$

We may also invoke the fact that the series $1+\sum_{n=1}^{\infty}\frac{(f(x))^n}{n!}$ is uniformly convergent so that we can integrate the series term by term. Note that for all x in [0, 1], $\left|\frac{(f(x))^n}{n!}\right| \le \frac{1}{n!}$ for $n > 1$ and $1+\sum_{n=1}^{\infty}\frac{1}{n!}$ is convergent. It follows by Weierstrass M test that $1+\sum_{n=1}^{\infty}\frac{(f(x))^n}{n!}$ is uniformly convergent on [0, 1] to the function g. As $\frac{(f(x))^n}{n!}$ is Riemann integrable for all $n \ge 1$, we can integrate $g(x)$ term by term. Next, we consider the integral $\int_0^1 \left(f(x)\right)^k dx$.

The Integral $\int_0^1 \left(f(x)\right)^k dx$.

We claim that $\int_0^1 \left(f(x)\right)^k dx = (-1)^k \frac{k!}{(k+1)^{k+1}}$.

$\int_t^1 \left(f(x)\right)^k dx = \int_t^1 x^k \left(\ln(x)\right)^k dx$. Let $u = -(k+1)\ln(x)$. Then $x = e^{-\frac{1}{k+1}u}$ and $\frac{du}{dx} = -(k+1)\frac{1}{x}$. Thus, applying a change of variable, we get

$$\int_t^1 \left(f(x)\right)^k dx = \int_t^1 x^k \left(\ln(x)\right)^k dx = -\int_{-(k+1)\ln(t)}^{0} e^{-u}(-1)^k \frac{1}{(k+1)^{k+1}} u^k du$$

$$= (-1)^k \frac{1}{(k+1)^{k+1}} \int_0^{-(k+1)\ln(t)} e^{-u} u^k du .$$

$$\int_0^1 \left(f(x)\right)^k dx = \lim_{t\to 0^+}\int_t^1 \left(f(x)\right)^k dx = \lim_{t\to 0^+}(-1)^k \frac{1}{(k+1)^{k+1}} \int_0^{-(k+1)\ln(t)} e^{-u} u^k du$$

$$= (-1)^k \frac{1}{(k+1)^{k+1}} \lim_{s\to\infty}\int_0^s e^{-u}u^k du = (-1)^k \frac{1}{(k+1)^{k+1}} \int_0^{\infty} e^{-u}u^k du \quad . \quad \text{---------- (1)}$$

Now we claim that $I_k = \int_0^{\infty} e^{-u}u^k du = k!$.

Note that $I_1 = \int_0^{\infty} e^{-u}u\,du = \lim_{s\to\infty}\int_0^s e^{-u}u\,du = \lim_{s\to\infty}\left[-e^{-u}u\right]_0^s + \lim_{s\to\infty}\int_0^s e^{-u}du$

$$= 0 + \lim_{s\to\infty}\left[-e^{-u}\right]_0^s = 1.$$

For $k > 1$, $I_k = \int_0^{\infty} e^{-u}u^k du = \lim_{s\to\infty}\int_0^s e^{-u}u^k du = \lim_{s\to\infty}\left[-e^{-u}u^k\right]_0^s + \lim_{s\to\infty} k\int_0^s e^{-u}u^{k-1}du$

$$= -\lim_{s\to\infty} e^{-s}s + kI_{k-1} = 0 + kI_{k-1} = kI_{k-1} \ .$$

It follows that $I_k = kI_{k-1} = k(k-1)I_{k-2} = \cdots = k!I_1 = k!$. Following (1) we get,

$$\int_0^1 \left(f(x)\right)^k dx = (-1)^k \frac{1}{(k+1)^{k+1}} \int_0^\infty e^{-u} u^k du = (-1)^k \frac{k!}{(k+1)^{k+1}} .$$

Therefore,

$$\int_0^1 x^x dx = \int_0^1 g(x)dx = \int_0^1 e^{f(x)} dx = 1 + \sum_{k=1}^\infty \int_0^1 \frac{(f(x))^k}{k!} dx = \sum_{k=0}^\infty (-1)^k \frac{1}{(k+1)^{k+1}} .$$

Similarly,

$$\int_0^1 x^{-x} dx = \int_0^1 e^{-f(x)} dx = 1 + \sum_{k=1}^\infty \int_0^1 \frac{(-f(x))^k}{k!} dx = \sum_{k=0}^\infty \frac{1}{(k+1)^{k+1}} .$$

This completes the derivation of the stated integrals.

Indeed, if a function $f(x)$ is Riemann integrable on the interval $[a, b]$, then it is bounded on $[a, b]$ and the function $e^{f(x)}$ is Riemann integrable on $[a, b]$. We may by the Arzelà's Dominated Convergence Theorem, integrate $e^{f(x)}$ on $[a, b]$ by term-by-term integration on the series expansion of $e^{f(x)} = 1 + \sum_{n=1}^\infty \frac{(f(x))^n}{n!}$ since the series is uniformly bounded by e^M , where $M = \max\left\{|f(x)| : x \in [a,b]\right\}$. That is,

$$\int_a^b e^{f(x)} dx = \sum_{k=0}^\infty \int_a^b \frac{(f(x))^k}{k!} dx .$$

Using the same approach, we can show that $\int_0^1 x^{(x^n)} dx = \sum_{k=0}^\infty (-1)^k \frac{1}{(nk+1)^{k+1}}$ and $\int_0^1 x^{-(x^n)} dx = \sum_{k=0}^\infty \frac{1}{(nk+1)^{k+1}}$.